Constructive Nonlinear Control of Underactuated Systems via Zero Dynamics Policies

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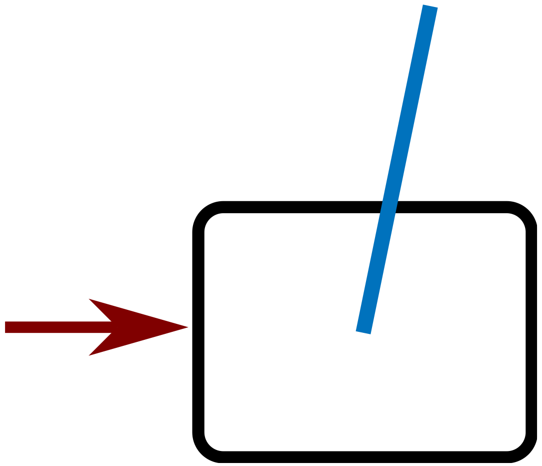
William D. Compton, Ivan D.J. Rodriguez, Noel Csomay-Shanklin,
Yisong Yue, Aaron D. Ames

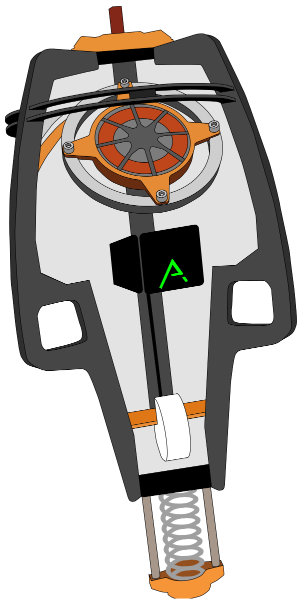
Definition: Underactuated

A system is *underactuated* if it has fewer actuators than degrees of freedom.



Actuated: $x, \ \dot{x}$

Unactuated: $\theta, \ \dot{\theta}$

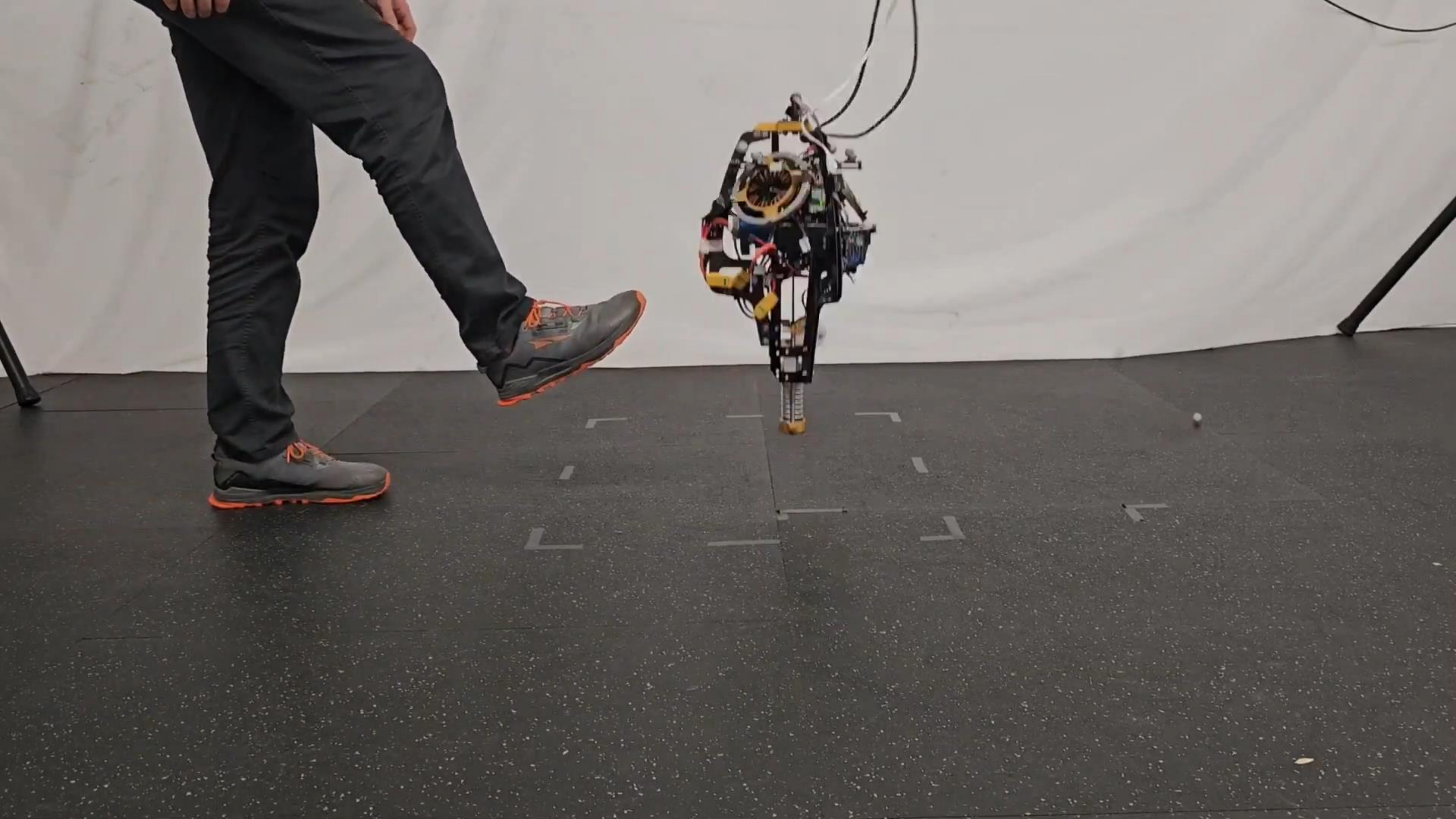


Actuated: $\bar{\mathbf{q}}, \ \boldsymbol{\omega}$

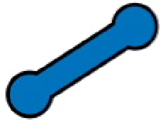
Unactuated: $x, y, \ \dot{x}, \dot{y}$

Actuated: $\mathbf{q}, \ \dot{\mathbf{q}}$

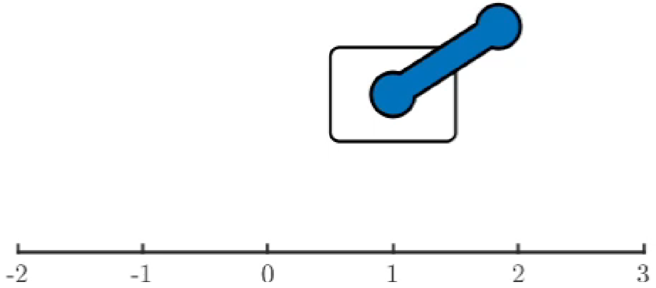
Unactuated: $\mathbf{x}_{com}, \ \dot{\mathbf{x}}_{com}$



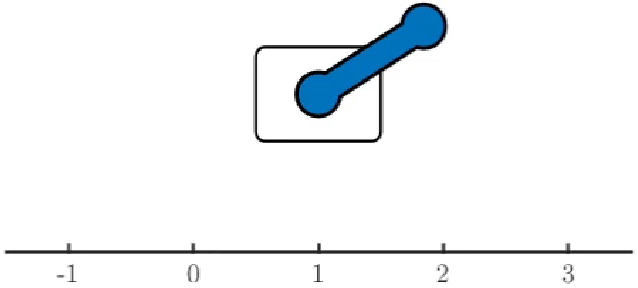
Pendulum: $y = \theta$



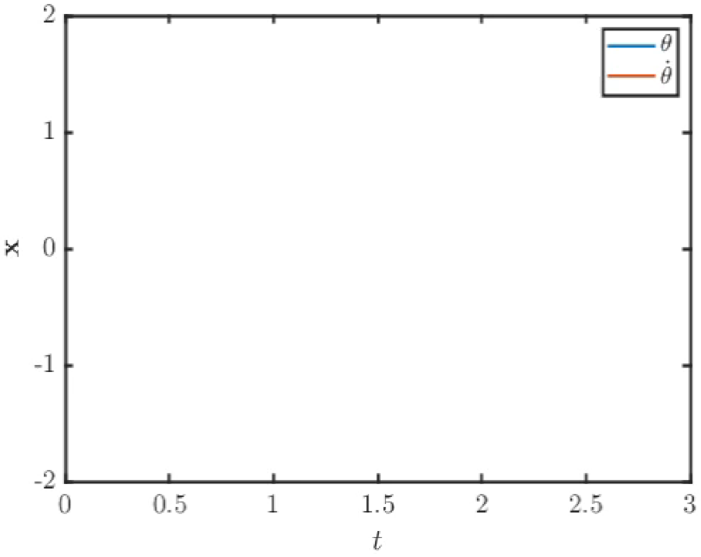
Cartpole: $y = x$

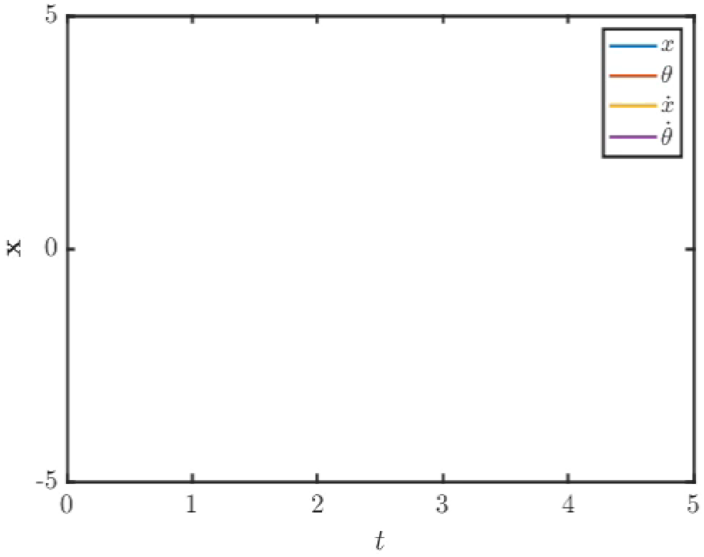


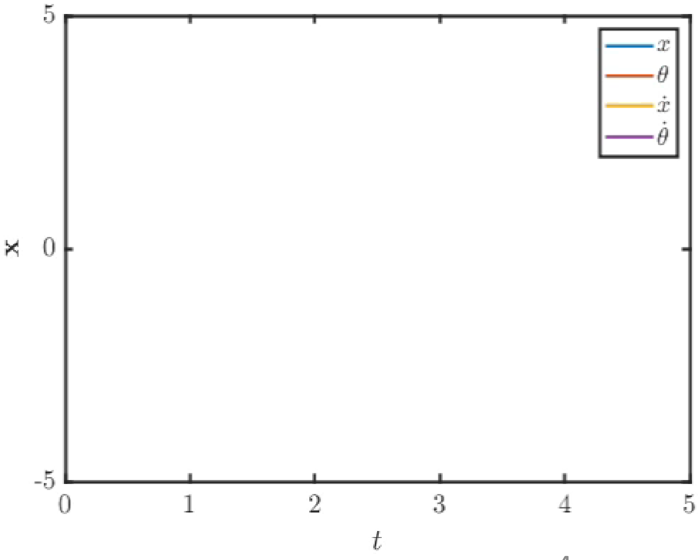
Cartpole: $y = \theta$



Can we constructively synthesize outputs $y = h(\mathbf{x})$, such that stabilizing the outputs results in stability of the full system state?







Definition: Actuation Decomposition (Normal Transform)

Given a system with state $\mathbf{x} \in \mathbb{R}^n$, input $v \in \mathbb{R}$, and output $y \in \mathbb{R}$ of *relative degree* γ,

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{g}_{\mathbf{x}}(\mathbf{x})v$$
$$y = h(\mathbf{x})$$

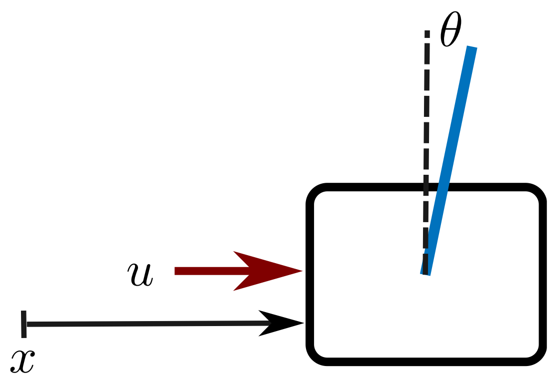
there exists a diffeomorphism $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^\gamma \times \mathbb{R}^{n-\gamma}$ mapping $\mathbf{x} \to (\boldsymbol{\eta}, \mathbf{z})$, and a feedback linearizing input $v = k_{fbl}(\mathbf{x}, u)$ such that the new coordinate system has dynamics:

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_\gamma \end{bmatrix} \qquad \begin{aligned} \dot{\eta}_1 &= \eta_2 \\ &\vdots \\ \dot{\eta}_{\gamma-1} &= \eta_\gamma \\ \dot{\eta}_\gamma &= u \end{aligned} \qquad \begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{F}\boldsymbol{\eta} + \mathbf{G}u \\ \dot{\mathbf{z}} &= \boldsymbol{\omega}(\boldsymbol{\eta}, \mathbf{z}) \\ y &= \eta_1 \end{aligned}$$

Exposes which states have direct actuation authority ($\boldsymbol{\eta}$) and which do not ($\mathbf{z}$).

For robotic systems, we can always choose y to be actuated joints.

Actuation Decomposition Example: Cartpole



Define Output Coordinates

$$\boldsymbol{\eta} = \begin{cases} y = x \\ \dot{y} = \dot{x} \end{cases}$$

$$\ddot{y} = a(\dot{x}, \theta, \dot{\theta}) + b(\dot{x}, \theta, \dot{\theta})v$$

Normal Coordinates Complete Diffeomorphism

$$\boldsymbol{\eta} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} \theta \\ m_p l(l\dot{\theta} + g\dot{x}\cos\theta) \end{bmatrix}$$

Feedback Linearizing Controller

$$v = \frac{1}{b(\dot{x}, \theta, \dot{\theta})}\left(-a(\dot{x}, \theta, \dot{\theta}) + u\right)$$

$$\ddot{y} = u$$

Actuation Decomposition

$$\dot{\boldsymbol{\eta}} = \mathbf{F}\boldsymbol{\eta} + \mathbf{G}u$$

$$\dot{\mathbf{z}} = \boldsymbol{\omega}(\boldsymbol{\eta}, \mathbf{z})$$

Proposed Output Structure and Zero Dynamics

$$\boldsymbol{\psi} : \mathcal{Z} \to \mathcal{N} \qquad \boldsymbol{\eta}_d = \boldsymbol{\psi}(\mathbf{z})$$

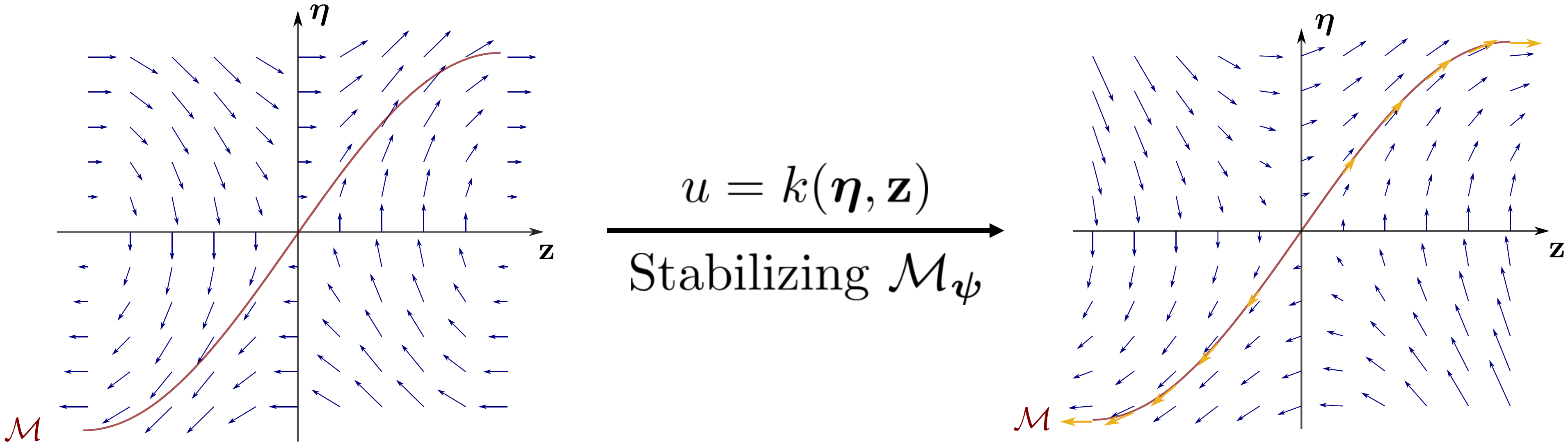
$\boldsymbol{\psi}$ maps underactuated coordinate to a desired actuated state, $\boldsymbol{\eta}_d$.

What is the structure of the points where $\boldsymbol{\eta} = \boldsymbol{\eta}_d$?

$$\mathbf{h}(\boldsymbol{\eta}, \mathbf{z}) = \boldsymbol{\eta} - \boldsymbol{\psi}(\mathbf{z}) = \mathbf{0}$$

Zeroing Manifold: $\mathcal{M}_{\boldsymbol{\psi}} = \{\boldsymbol{\eta}, \mathbf{z} \mid \mathbf{h}(\boldsymbol{\eta}, \mathbf{z}) = \mathbf{0}\}$

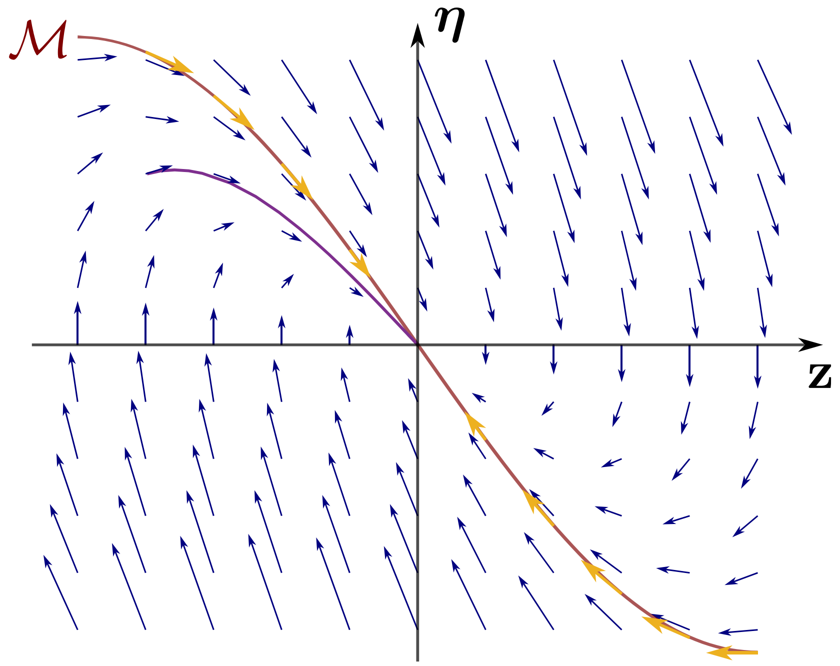
Lemma 2. *If $\mathcal{M}_{\boldsymbol{\psi}}$ is controlled invariant, then $\mathcal{M}_{\boldsymbol{\psi}}$ is the zeroing manifold associated with the output $y = \eta_1 - \psi_1(\mathbf{z})$.*



Zero Dynamics: $\dot{\mathbf{z}} = \boldsymbol{\omega}(\boldsymbol{\psi}(\mathbf{z}), \mathbf{z})$ (Dynamics on $\mathcal{M}_{\boldsymbol{\psi}}$)

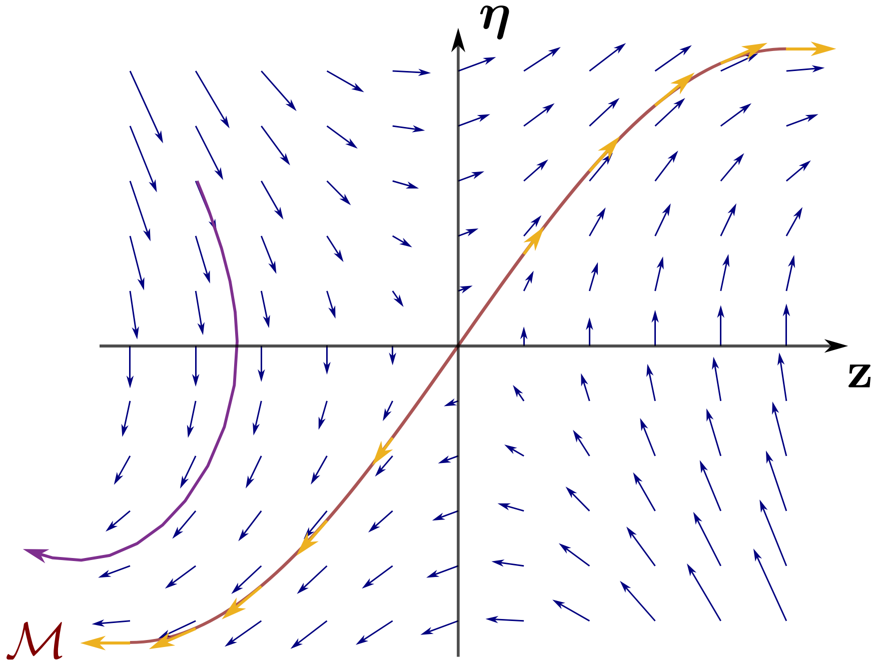
Theorem 1: Composite Stability

If the zero dynamics $\dot{\mathbf{z}} = \boldsymbol{\omega}(\boldsymbol{\psi}(\mathbf{z}), \mathbf{z})$ are stable, then stabilizing the manifold $\mathcal{M}_{\boldsymbol{\psi}}$ by driving $\mathbf{h}(\boldsymbol{\eta}, \mathbf{z}) \rightarrow \mathbf{0}$ results in stability of the entire system, i.e. $\mathbf{x} \rightarrow \mathbf{0}$.



Desirable Properties of $\mathcal{M}_{\boldsymbol{\psi}}$

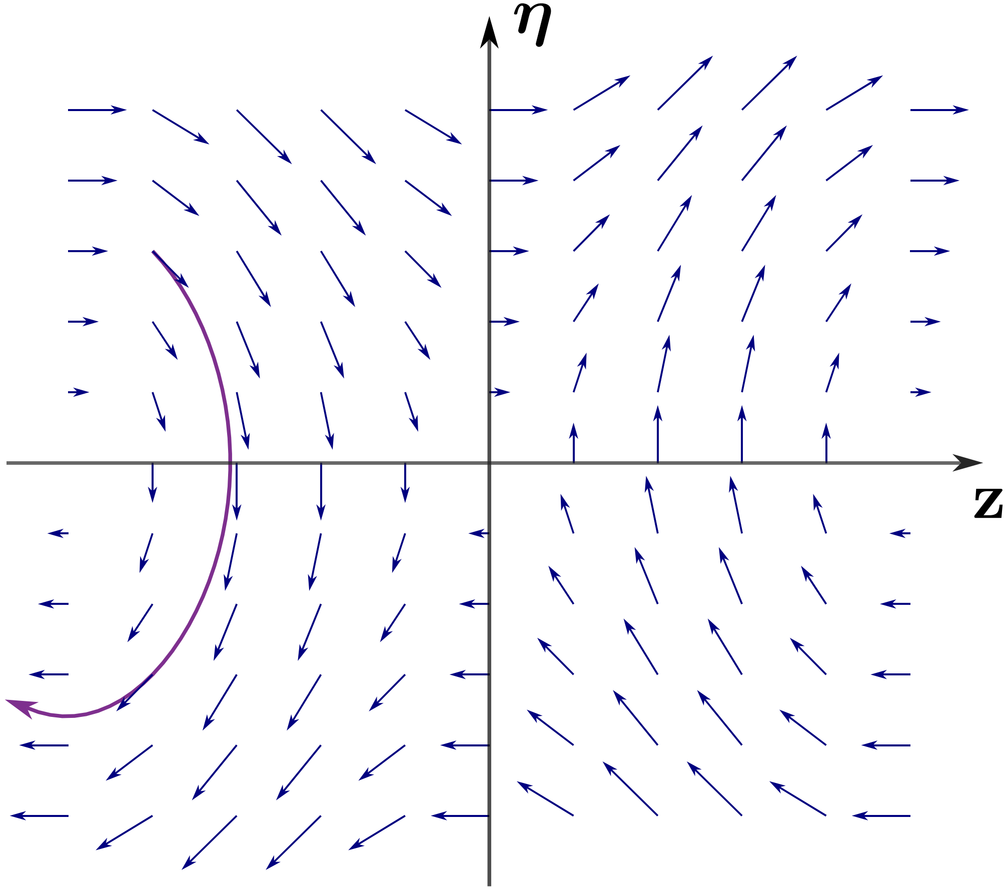
1. Able to be rendered attractive.
   True as long as it is controlled invariant by Lemma 2 (benefit of actuation structure).
2. Stable autonomous dynamics.
   Completely determined by choice of $\boldsymbol{\psi}$.



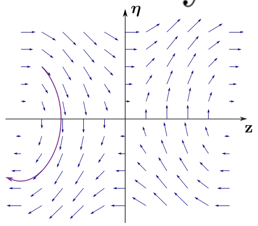
Question: When does a manifold with these desirable properties exist?

Always. (around equilibria, for controllable systems).
In fact, we can construct them from the linearization.

Visual Example:



(Nominal Dynamics)

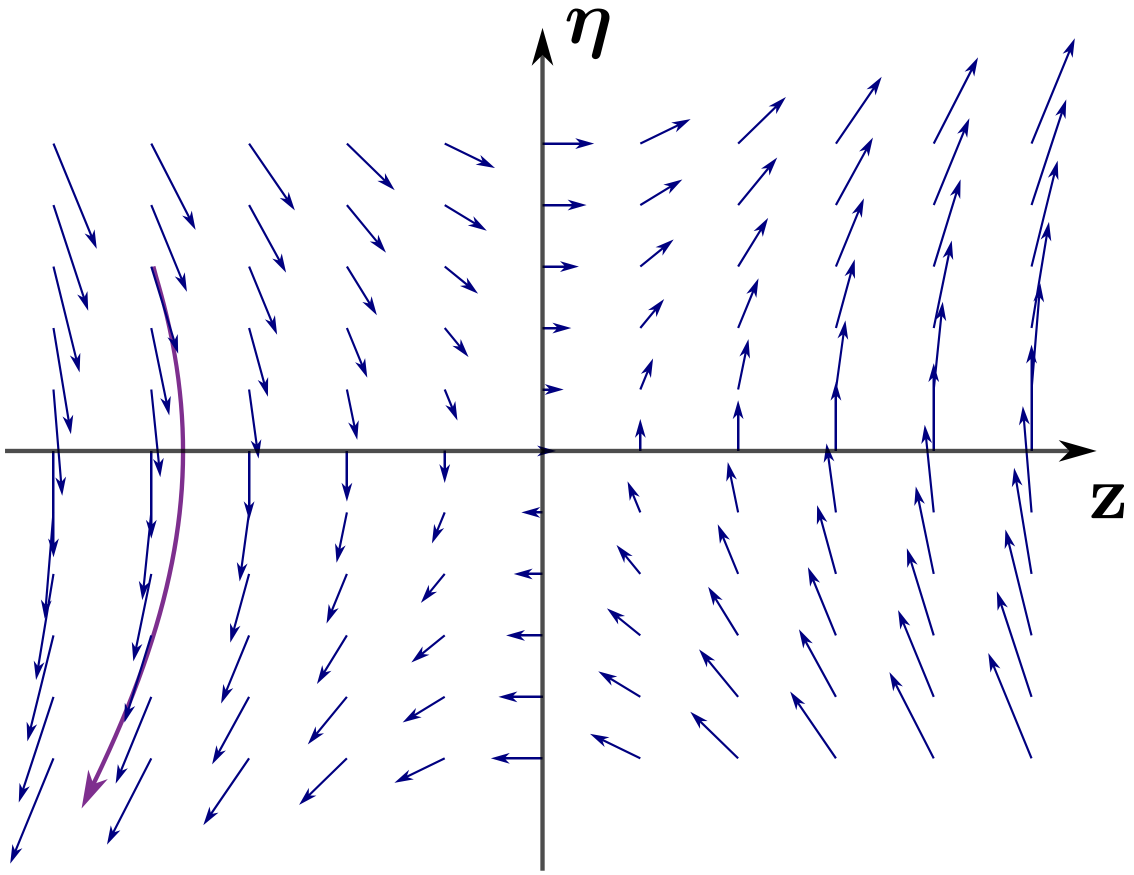


1. Linearize

2. Control Linearization

3. Identify $\mathcal{S}$

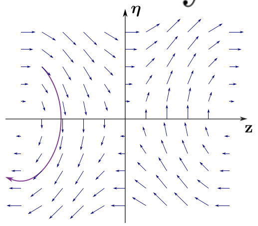
4. Stabilize $\mathcal{M}$ for NL dyn.



$$\dot{\boldsymbol{\eta}} = \mathbf{F}\boldsymbol{\eta} + \mathbf{G}u$$
$$\dot{\mathbf{z}} = \boldsymbol{\omega}(\boldsymbol{\eta}, \mathbf{z})$$

$$\rightarrow \quad \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{A}_{\eta} & \mathbf{A}_{\mathbf{z}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix} u$$

(Nominal Dynamics)



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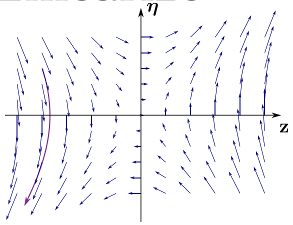

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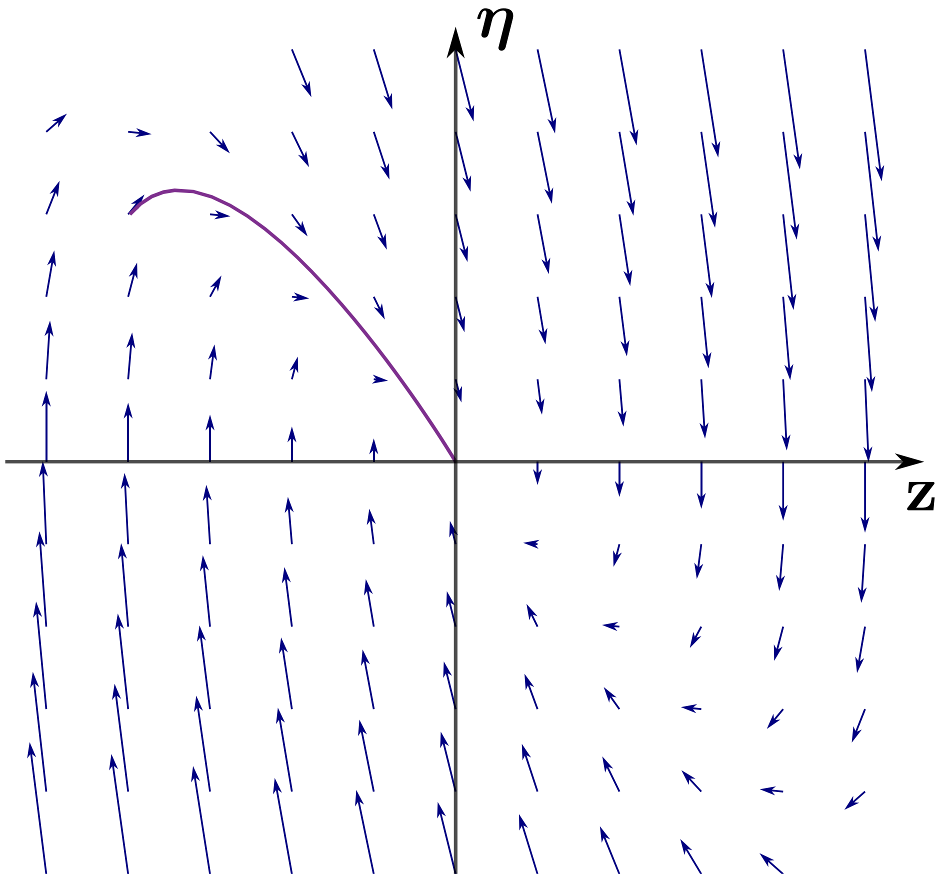
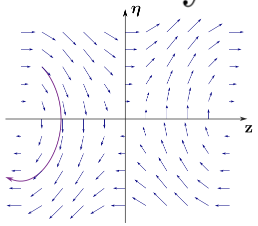


Controllable by assumption $\implies$ LQR, Pole Placement

(Nominal Dynamics)



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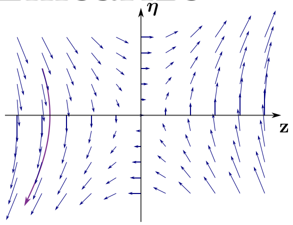
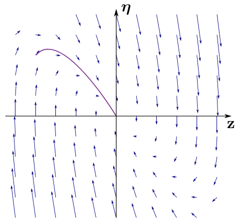


2. Control Linearization

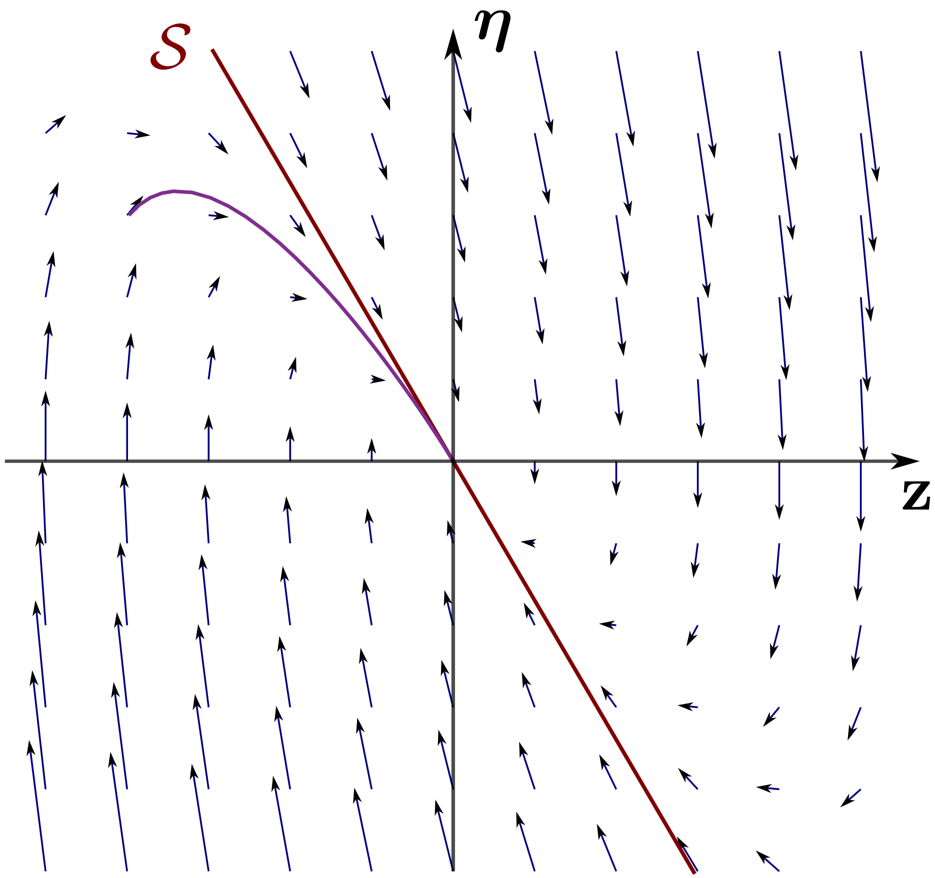


3. Identify $\mathcal{S}$

4. Stabilize $\mathcal{M}$ for NL dyn.

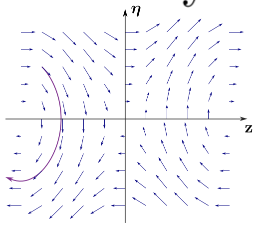
$$\mathcal{S} = \{(\boldsymbol{\eta}, \mathbf{z}) \mid \boldsymbol{\eta} = \mathbf{S}\mathbf{z}\}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_{\eta_1}^\top \\ \vdots \\ \mathbf{s}_{\eta_\gamma}^\top \end{bmatrix}$$



Lemma 3. *There exists a nonempty set of controllers which stabilize the linearized system and induce an n_z dimensional invariant subspace $\mathcal{S}$ such that for each $\mathbf{z}$ there exists a unique $\boldsymbol{\eta}$ such that $(\boldsymbol{\eta}, \mathbf{z}) \in \mathcal{S}$.*

(Nominal Dynamics)



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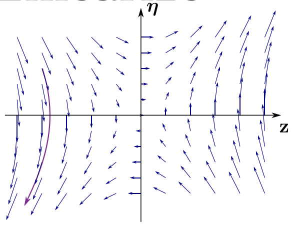

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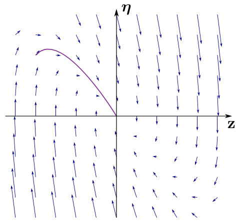

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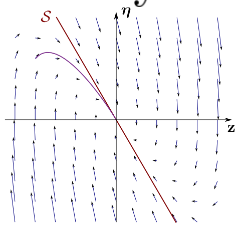
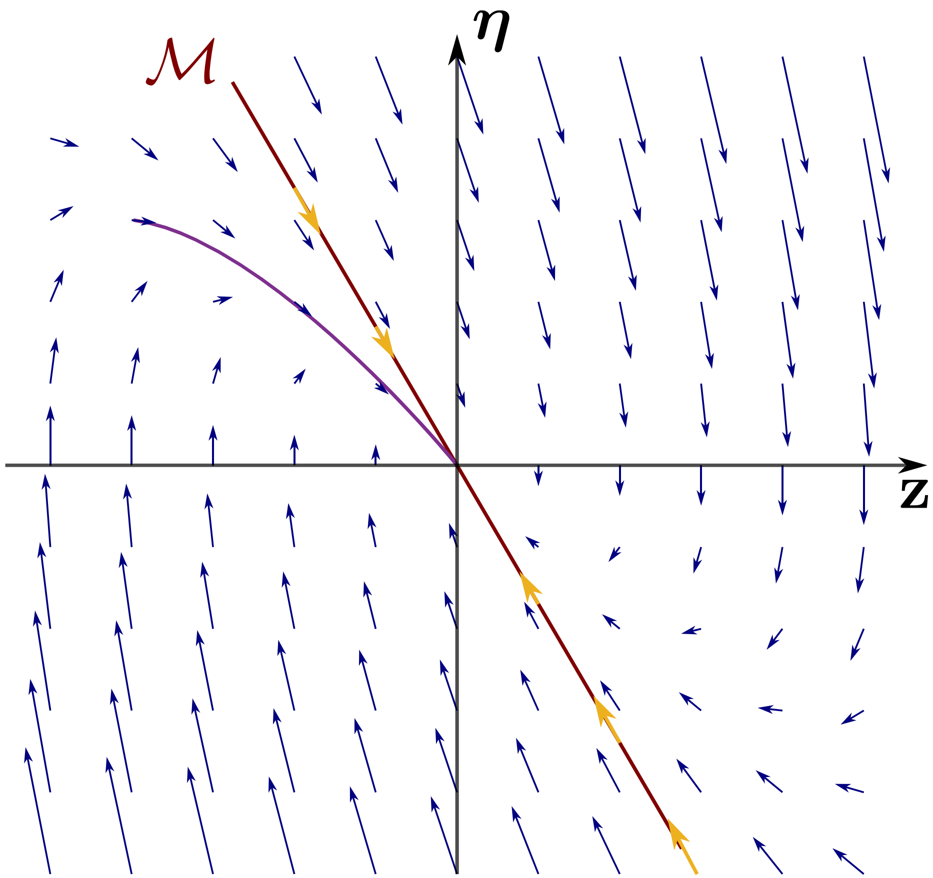


4. Stabilize $\mathcal{M}$ for NL dyn.

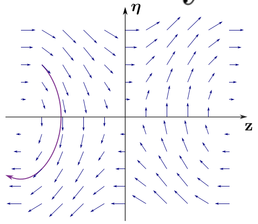
$$\mathcal{S} = \{(\boldsymbol{\eta}, \mathbf{z}) \mid \boldsymbol{\eta} = \mathbf{S}\mathbf{z}\}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_{\eta_1}^\top \\ \vdots \\ \mathbf{s}_{\eta_\gamma}^\top \end{bmatrix}$$



Theorem 2. *Given a nonlinear system, the output* $y = \eta_1 - s_{\eta_1}^\top z$ *obtained via linearization and Lemma 3 has valid relative degree and exponentially stable zero dynamics for the nonlinear system. As such, stabilizing* $\mathbf{e} \to \mathbf{0}$ *results in stability of the entire system to the origin.*

(Nominal Dynamics)



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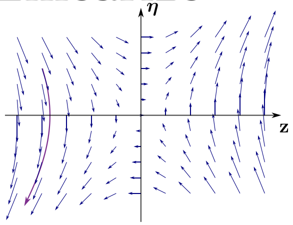

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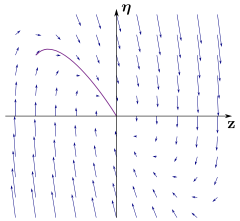

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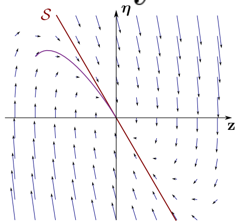
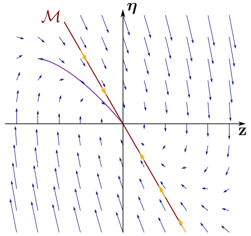


4. Stabilize $\mathcal{M}$ for NL dyn.



Theorem 2: Stabilizing Output Construction

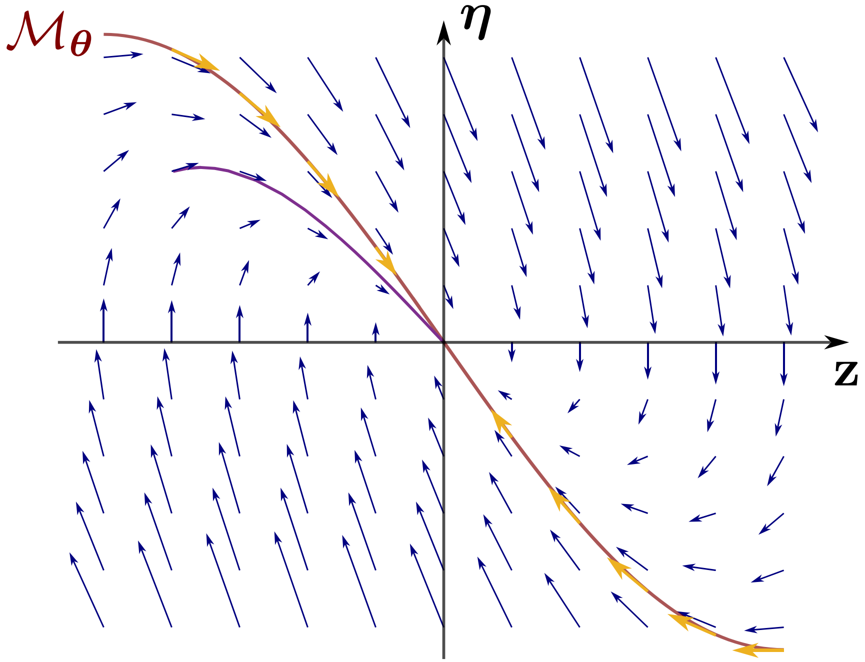
Given a nonlinear system, the output $y = \eta_1 - \mathbf{s}_{\eta_1}^\top \mathbf{z}$, obtained via linearization and Lemma 3, has valid relative degree and exponentially stable zero dynamics for the nonlinear system. As such, zeroing the output results in stability of the entire system.

Proof Approach:

- $y = \eta_1 - \mathbf{s}_{\eta_1}^\top \mathbf{z}$ has valid relative degree γ for the linear system.
- Driving $y \to 0$ drives the system to $(\boldsymbol{\eta}, \mathbf{z}) \to \mathcal{S}$.
- Close to the origin, both the relative degree property and stability of the Zero Dynamics can be transfered to the nonlinear system by bounding the deviation of the dynamics from the linear system.

Using learning to extend the region of validity of Zero Dynamics Policies

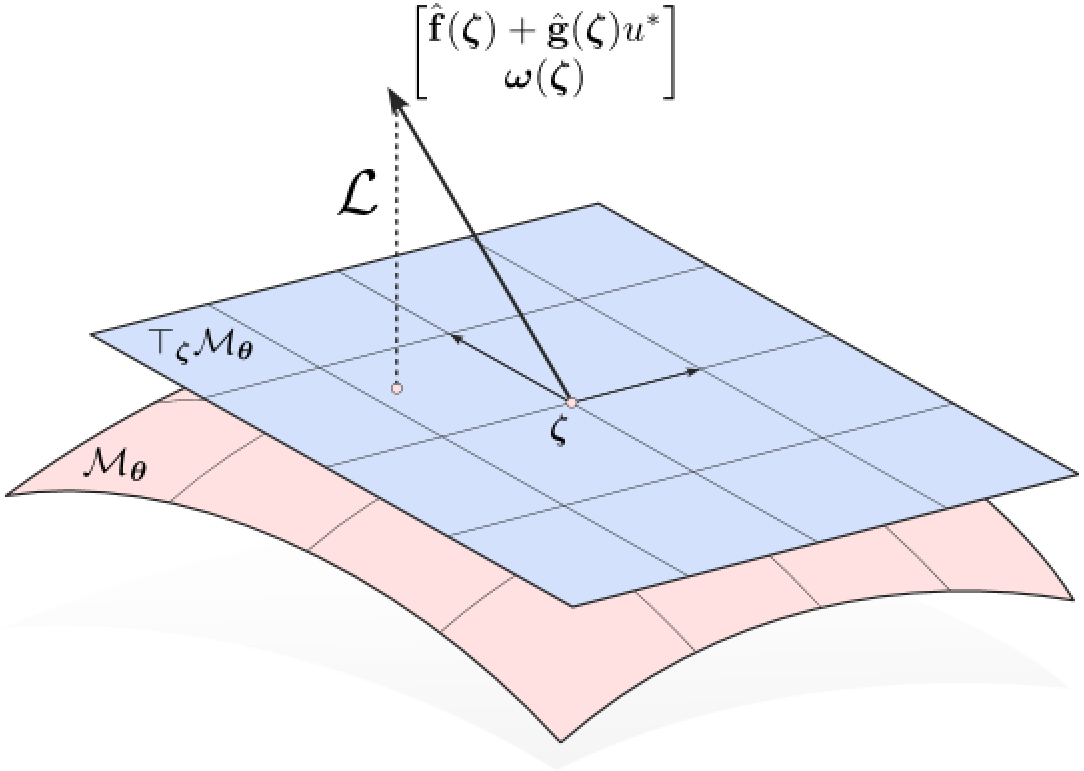
Parameterize the ZDP as a neural network, $\boldsymbol{\psi}_{\boldsymbol{\theta}}(\mathbf{z})$.



Training: Optimize $\boldsymbol{\theta}$ such that $\mathcal{M}_{\boldsymbol{\theta}}$ is *invariant under stabilizing trajectories.*

$$\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z}\sim Z}\left\|\hat{\mathbf{f}}(\boldsymbol{\zeta}) + \hat{\mathbf{g}}(\boldsymbol{\zeta})u^* - \frac{\partial \boldsymbol{\psi}_{\boldsymbol{\theta}}}{\partial \mathbf{z}}\boldsymbol{\omega}(\boldsymbol{\zeta})\right\|^2 \qquad \boldsymbol{\zeta} = (\boldsymbol{\psi}_{\boldsymbol{\theta}}(\mathbf{z}), \mathbf{z})$$

$$\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z}\sim Z}\left\|\hat{\mathbf{f}}(\boldsymbol{\zeta}) + \hat{\mathbf{g}}(\boldsymbol{\zeta})u^* - \frac{\partial \boldsymbol{\psi}_{\boldsymbol{\theta}}}{\partial \mathbf{z}}\boldsymbol{\omega}(\boldsymbol{\zeta})\right\|^2 \qquad \boldsymbol{\zeta} = (\boldsymbol{\psi}_{\boldsymbol{\theta}}(\mathbf{z}), \mathbf{z})$$

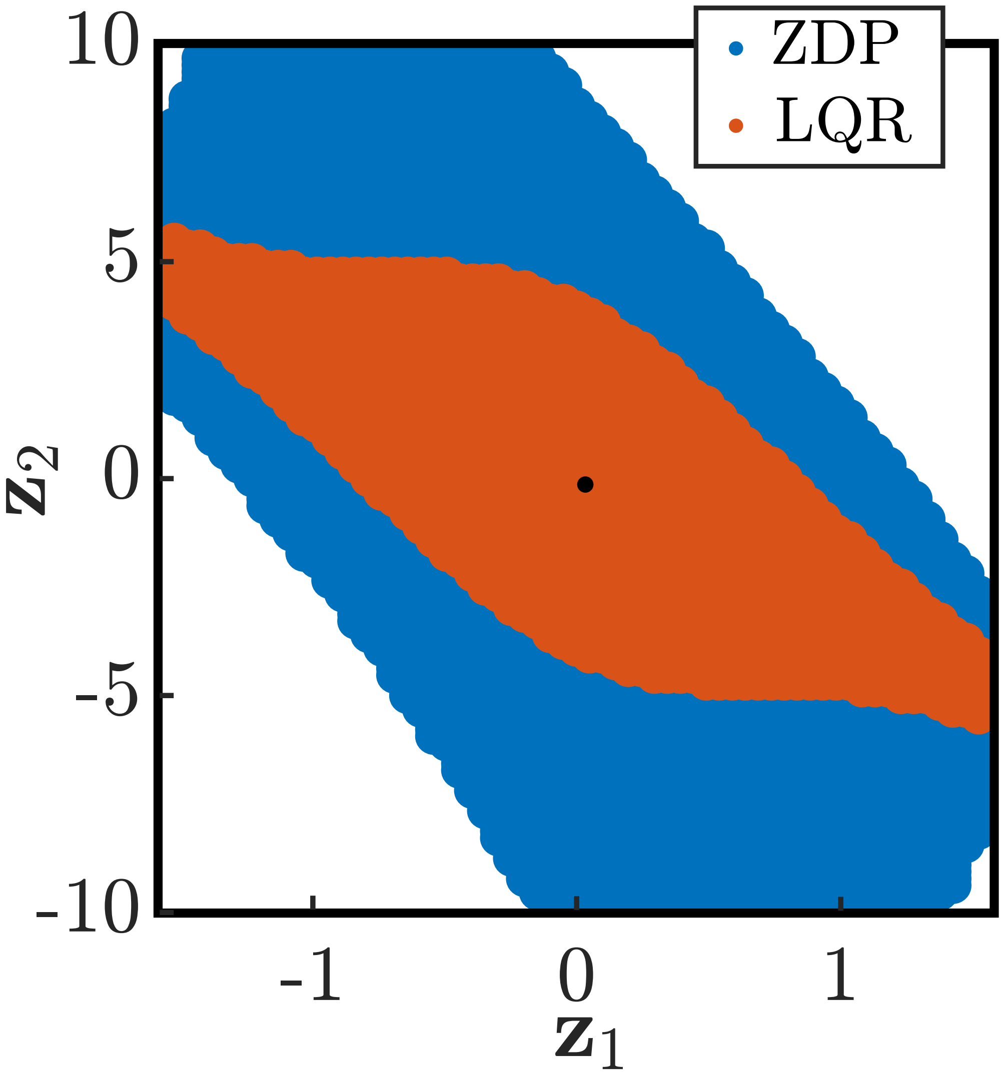


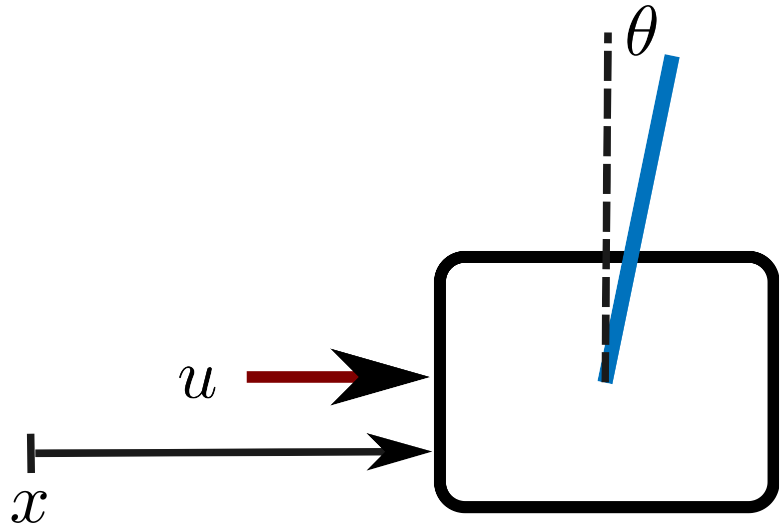
We have developed constructive methods for synthesizing stablizing outputs for underactuated systems.

Leverage linearization of the system about equilbrium (and linear control) for local synthesis (and proof of existence).

Learning problem which leverages optimal control to find manifolds $\mathcal{M}_{\boldsymbol{\theta}}$ with desirable properties.

Region of Attraction for (Unstable) Cartpole

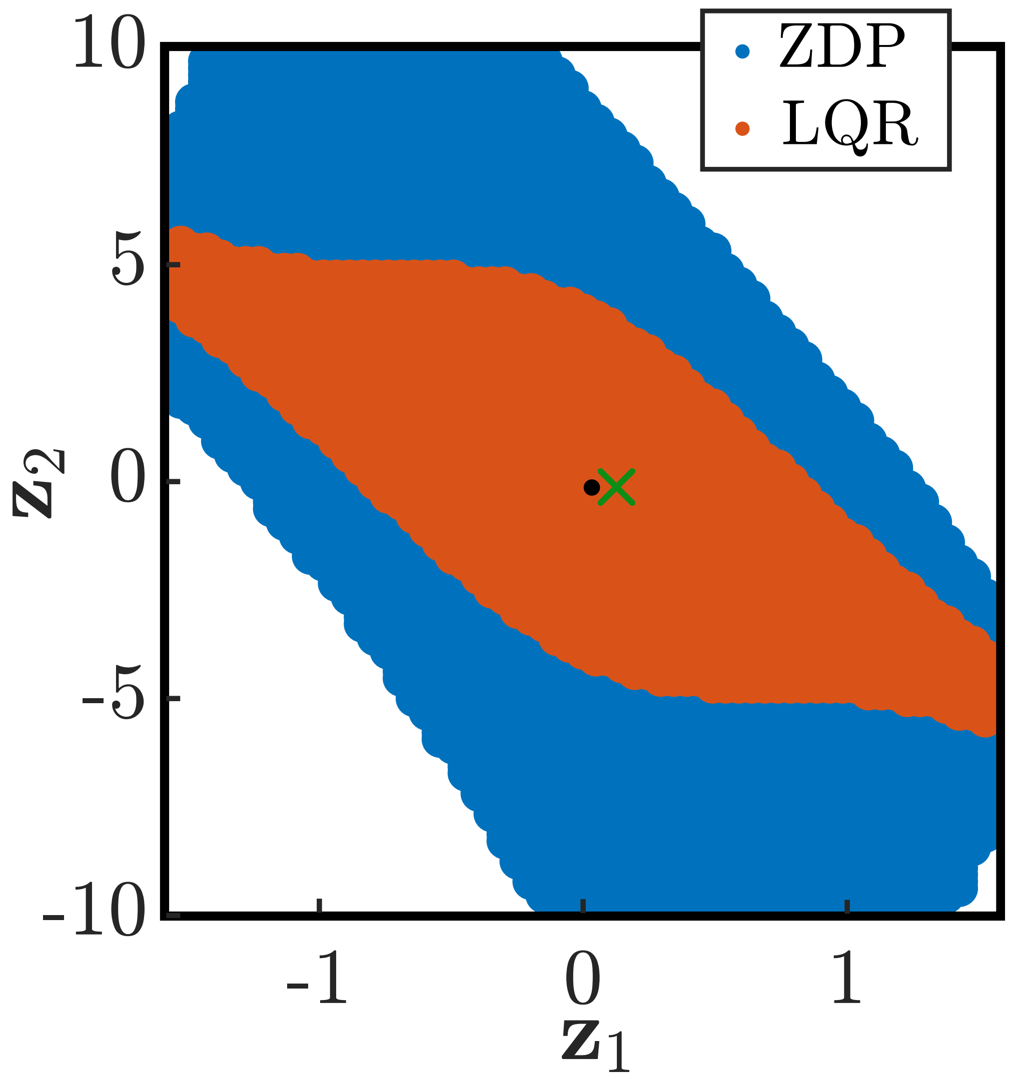


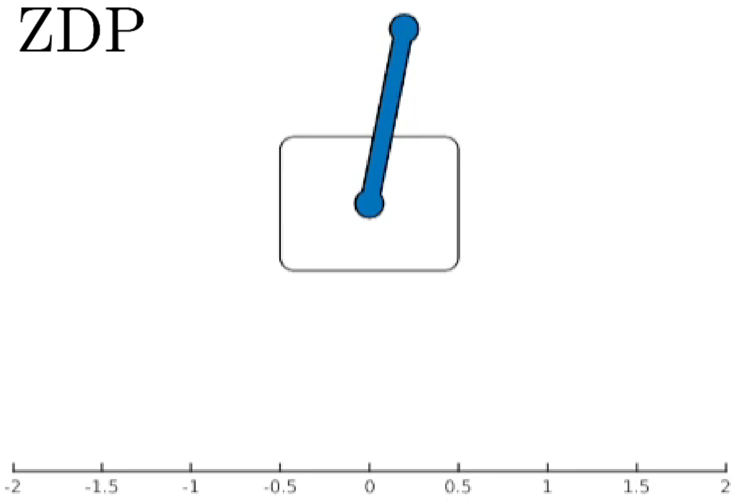


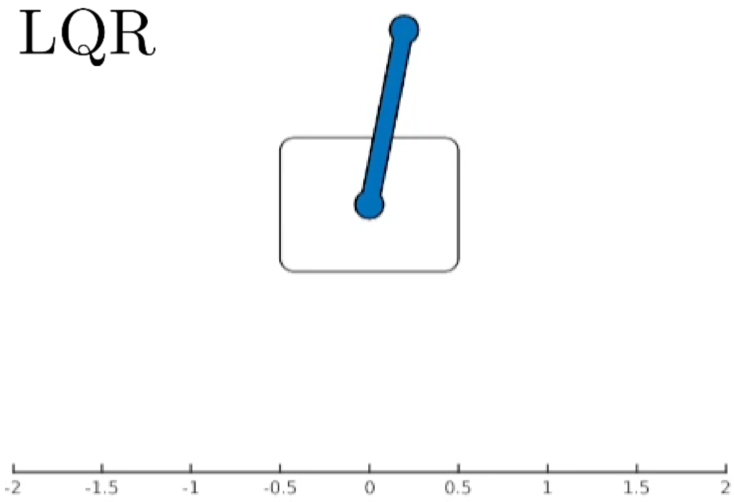
$$z_1 = \theta$$

$$z_2 = m_p l(l\dot{\theta} + g\dot{x}\cos\theta)$$

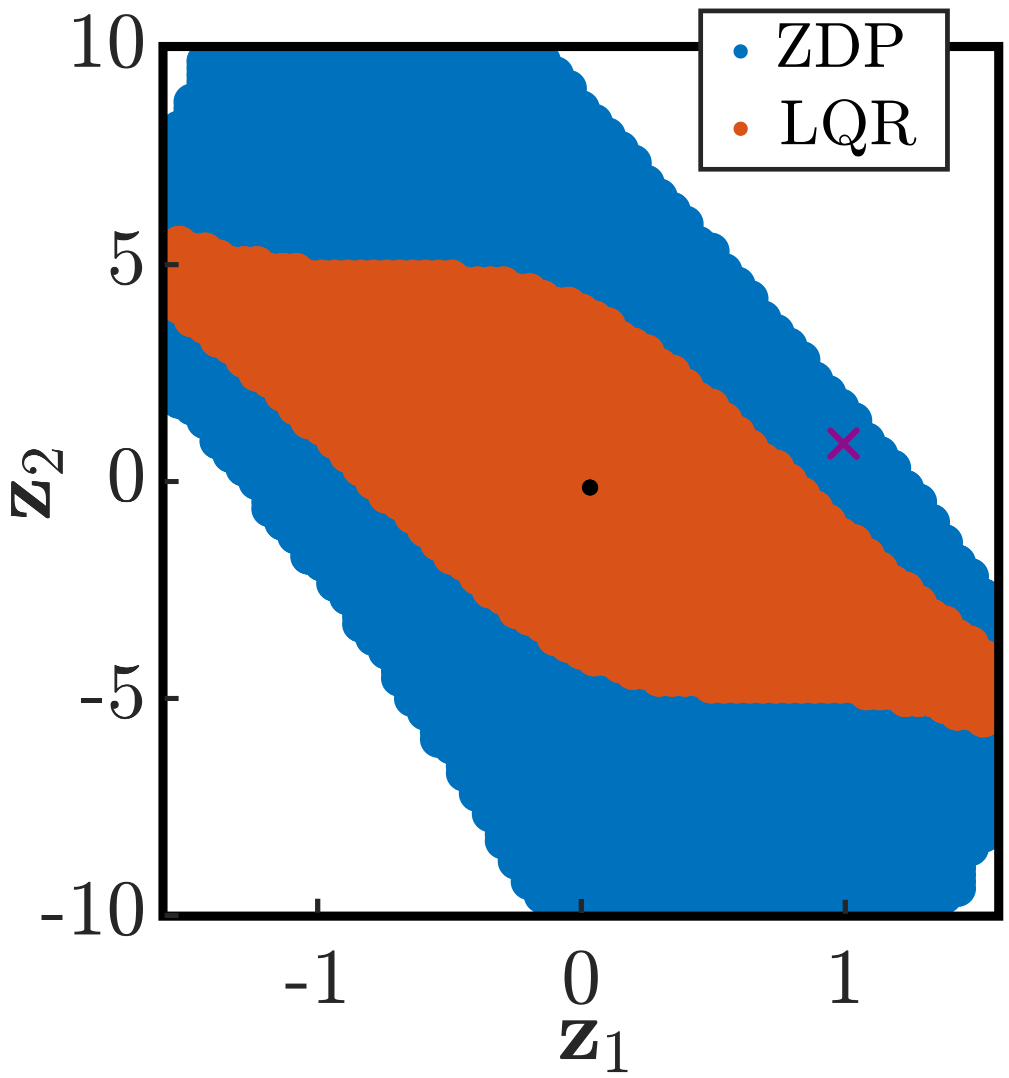
LQR vs. ZDP Response Comparison

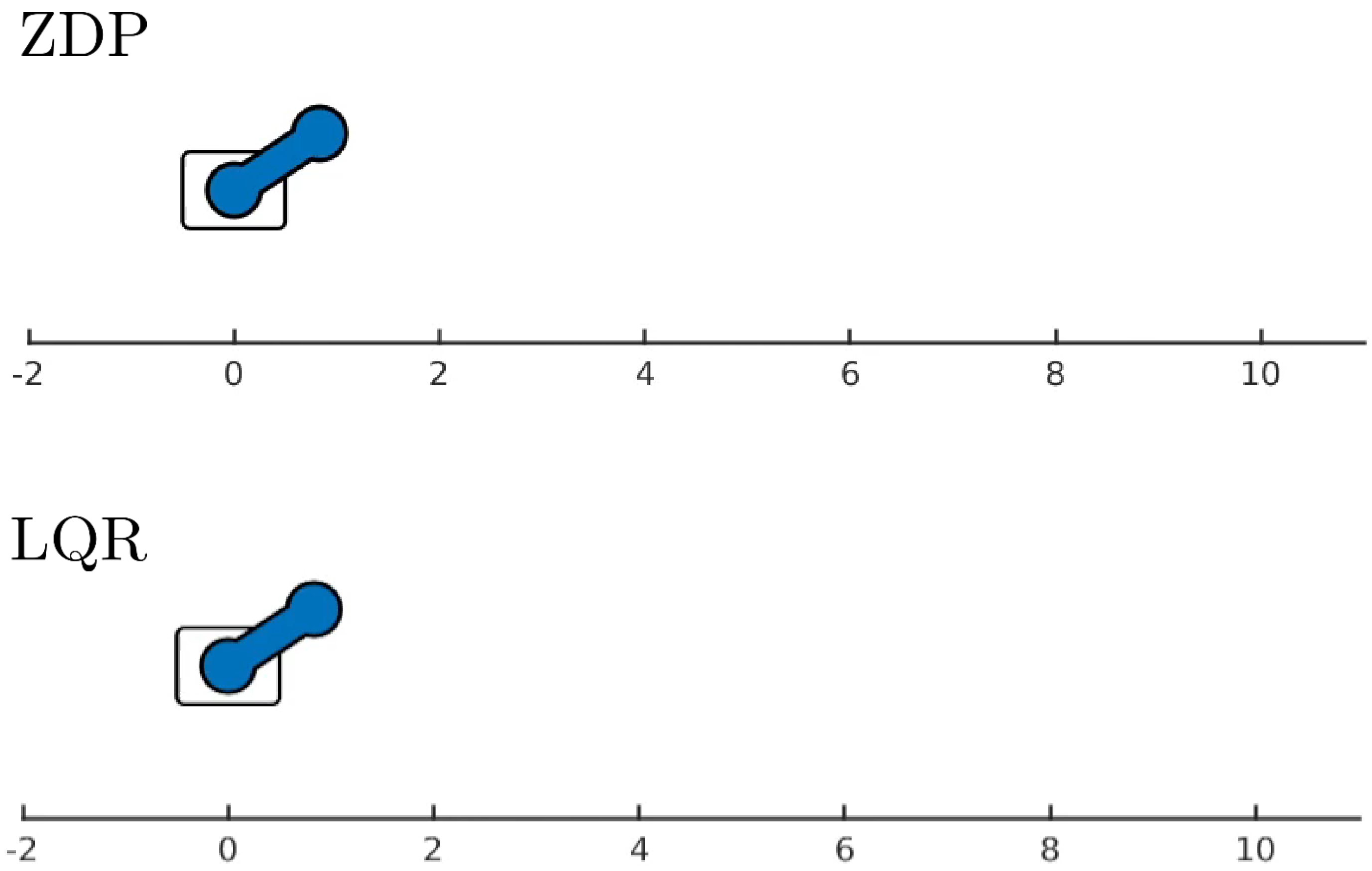






LQR Failure - ZDP Successful





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Questions?

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